AD 633282

Carnegie Institute of Technology

Pittsburgh 13, Pennsylvania

DDC
RETURNED
JUN 6 1966
C

CLEARINGHOUSE
FOR FEDERAL SCIENTIFIC AND
TECHNICAL INFORMATION

Hardcopy	Microfiche		
$1.00	$.50	12 pp	00

ARCHIVE COPY

Code 1

GRADUATE SCHOOL of INDUSTRIAL ADMINISTRATION

William Larimer Mellon, Founder

Management Sciences Research Report No. 76

A CONVEX APPROXIMANT METHOD FOR NON-CONVEX EXTENSIONS OF GEOMETRIC PROGRAMMING

by

A. Charnes
Northwestern University
and
W. W. Cooper
Carnegie Institute of Technology

May 5, 1966

MANAGEMENT SCIENCES RESEARCH GROUP
GRADUATE SCHOOL OF INDUSTRIAL ADMINISTRATION
CARNEGIE INSTITUTE OF TECHNOLOGY
PITTSBURGH, PENNSYLVANIA 15213

This report was prepared as part of the activities of the Management Sciences Research Group, Carnegie Institute of Technology, (Under Contract NONR 760(24), NR 047-048 with the U. S. Office of Naval Research) and as part of the activities of the Systems Research Group, Northwestern University (under Contract NONR 1228(10), NR 047-021 with the U. S. Office of Naval Research), and also with the Department of the Army Contract No. DA-31-124-AROD-322. Distribution of this document is unlimited. Reproduction of this paper in whole or in part is permitted for any purpose of the United States Government.

1. Introduction:

Many important problems of engineering and management are of a form which could be represented as geometric programs except that the functional to be minimized as well as the constraints are not confined to "posynomials"[1/] in that some of the coefficients are negative. The resulting problem thus may not, in general, be transformed to an equivalent convex programming problem.[2/] To date the only general method for obtaining global optima to (necessarily non-convex) problems with multiple local optima is Gomory's integer programming method.[3/]

We are herewith proposing an approximate method for another class of problems with multiple local optima--viz., extensions of geometric programming in which some of the coefficients are negative. This method provides, at each stage, a convex approximant which, a fortiori, provides the duality relations that are needed for many purposes. This is in contrast to other approaches which either lose these duality relations[4/] or else restrict the applications to special situations.[5/] More specifically,

---

1/ Cf. [7] for definitions of this and other terminology in geometric programming.

2/ Cf., e.g., the exponential transformations used in [3] and [4].

3/ See [8] and [9] for Gomory's original articles. See also [2] and [6] for further discussion and development.

4/ Cf., e.g., [10].

5/ The constraints in [3] and [5] , for instance, were arranged so that they could always be treated in a manner which did not preclude access to the indicated duality. Other possibilities are also present, however, as witness some of the examples, treated in [7].

the method that we shall describe here is conceived in the same spirit as previous suggestions we have made as a result of other research we have conducted to extend the boundaries of ordinary linear programming.[1/]

2. Formulation and Development of the Convex Approximant:[2/]

Consider the following problem

$$\begin{aligned} &\min \quad g_o^+ - g_o^- \\ &\text{subject to} \\ &\qquad g_i^+ - g_i^- \leq 1, \ i=1, \ldots, m \end{aligned} \tag{1.3}$$

where the g_k^+, g_k^- are posynomials in

$$t = (t_1, \ldots, t_n) . \tag{1.2}$$

I.e.,

$$g_i^+ = \sum_{j \epsilon J_i} P_{ij}^+(t) \ ; \quad g_i^- = \sum_{k \epsilon K_i} P_{ij}^-(t)$$

$$\begin{aligned} P_{ij}^+(t) &= c_{ij}^+ \, t_1^{a_1^{ij}} \cdot \ldots \cdot t_n^{a_n^{ij}} \\ P_{ij}^-(t) &= c_{ij}^- \, t_1^{b_1^{ij}} \cdot \ldots \cdot t_n^{b_n^{ij}} \end{aligned} \tag{1.3}$$

$$c_{ij}^+, \ c_{ij}^- > 0 .$$

1/ Cf., e.g., [1] and [5].

2/ To abbreviate this part of the development, it is assumed that all conditions for existence and attainment of the indicated minima are fulfilled. Cf. [7] for a rigorous treatment of the relevant necessary and sufficient conditions in complete detail.

Note that the above problem is a generalization of ordinary geometric programming in that the constraints and the functional are not confined to posynomials.

3. Formulation of Approximants:

Each one-term posynomial $P_{ij}^{-}(t)$ in the preceding expressions may be replaced by a single variable y_{ij} subject to

$$y_{ij} \leq P_{ij}^{-}(t) \tag{2.1}$$

or

$$y_{ij} \, [P_{ij}^{-}(t)]^{-1} \leq 1 \tag{2.2}$$

which is the same as

$$\frac{y_{ij}}{c_{ij}^{-}} \left[t_1^{-b_1^{ij}} \cdots t_n^{-b_n^{ij}} \right] \leq 1 . \tag{2.3}$$

The resulting problem in t and the y_{ij} is equivalent to (1.1).

Next, let us suppose that the range of each y_{ij} relevant to the optimization may be represented by

$$0 < L_{ij} \leq y_{ij} \leq U_{ij} \tag{3}$$

We then introduce $k_{ij} > U_{ij}$ and consider the function

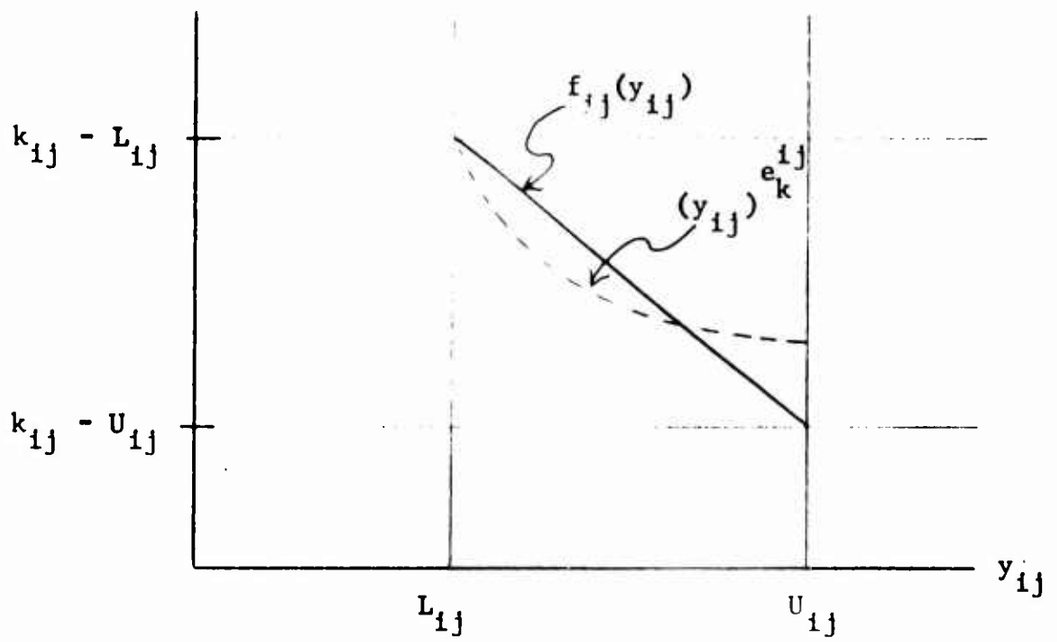
$$f_{ij}(y_{ij}) = k_{ij} - y_{ij} \tag{4}$$

as diagrammed below. Evidently over the interval (L_{ij}, U_{ij}) the linear function (4) is positive and bounded above and below. It may thus be

approximated by a posynomial

$$q_{ij}(y_{ij}) = \sum_k d_k^{ij} (y_{ij})^{e_k^{ij}} \tag{5}$$

where the d_k^{ij} are suitably selected positive constants.



To the degree of approximation thus rendered--e.g., approximation of the linear function by posynomials--the original problem (1.1) is now replaced by

$$\min \; g_o^+(t) + \sum_{j=1}^{m_o} q_{oj}\,(y_{oj})$$

subject to

$$g_i^+(t) + \sum_{j=1}^{m_o} q_{ij}\,(y_{ij}) \le 1 + \sum_j k_{ij} \tag{6}$$

$$[P_{ij}^-(t)]^{-1}\, y_{ij} \le 1$$

$$y_{ij}\, U_{ij}^{-1} \le 1$$

$$y_{ij}^{-1}\, L_{ij} \le 1$$

$$t > 0$$

This problem may evidently be transformed (e.g., by the exponential transformation)[1/] into a convex programming problem. We therefore call it a convex approximant of the original problem. It therefore follows that it has only one local (= global) optimum value.

Note in particular that each convex approximant has an associated dual problem. Thus a dual evaluator is available for each constraint. Those that refer to the U_{ij}, L_{ij} constraints indicate possible directions of improvement if these upper or lower bounds are tight. The dual evaluator is, of course, equal to zero when these bounds are slack. The approximation can thus be improved in the neighborhood of any already attained optimum by, e.g., reducing the range of the slack U_{ij} and L_{ij}, thereby enabling one to make an improved posynomial fit in the next

---

1/ See [3] and [4].

convex programming approximant. Similarly, the interval may be reduced and translated in the direction indicated by the non-zero dual evaluator for the tight U_{ij}, L_{ij} constraints.

Thus, sequentially, the convex approximant can be refined. One would expect the global optimum to be obtained by this method in situations where the original problem has multiple local optima. For, if the global optimum value were significantly different from that of other local optima, one would anticipate that the small modifications of the smooth continuous functions to equally smooth continuous approximants would not significantly alter the global optimum. Since the convex approximant has only one local (= global) optimum, its value should therefore be close to the global optimum value of the original problem. On the other hand, when the global optimum value of the original does not differ significantly fromother local optimum values, the precise optimum obtained matters little so far as value is concerned. In either situation therefore one would expect a sequence of convex approximants to yield a worthwhile result.

3. Conclusion:

In the paper [4], we showed how geometric programming could be applied to the determination of multiple simultaneous EOQ (economic order quantity) formulas under constraints as well as to aspects of the economic theory of production (e.g., with Cobb-Douglas and generalized SMAC production functions). Still further extensions in this direction

(e.g., to problems of capital budgeting) critically depend on the possibility of dealing with the presence of negative coefficients--as in (1.1)--and the same is true even of the originally motivated applications to engineering designs when, for instance, scrap values require consideration. Even more important, however, is the need for increased flexibility as when, for instance, there is a need to deal with problems where the natural original orientation is toward maximization (rather than minimization) and where a restriction to posynomials only makes it impossible to proceed through the negative of an associated minimization problem. 1/ A recourse to the convex approximant method would then seem to be in order--at least in these cases and possibly others as well.

1/ E.g., as in ordinary linear programming. Cf., e.g., [2] or [6].

BIBLIOGRAPHY

[1] Charnes, A., and W. W. Cooper, "Elements of a Strategy for Making Models in Linear Programming," Ch. 26 in R. Machol, et. al., eds. System Engineering Handbook (New York: McGraw-Hill Book Co., Inc., 1965).

[2] _____ _____, Management Models and Industrial Applications of Linear Programming (New York: John Wiley & Sons, Inc., 1961).

[3] _____ _____, "Optimizing Engineering Designs under Inequality Constraints," Cahiers du Centre d'Etudes de Recherche Operationnelle 6, No. 4., 1964, pp. 181-187. Also presented in "Foundations and Tools for Operations Research and the Management Sciences" (Ann Arbor: University of Michigan, June, 1962).

[4] _____ _____ and K. Kortanek, "An Alternative Approach to Duality with Suggested Applications to Economics and Management Science" Evanston, Ill., Northwestern University, The Technological Institute Systems Research Group Report No. and Pittsburgh, Pa., Carnegie Institute of Technology, Management Sciences Research Group Report No. 65, December, 1965.

[5] _____ _____ and E. Schatz "Management Mathematics and Engineering Designs," Proc. Second Conf. Engineering Designs (Los Angeles: University of California, August, 1962).

[6] Dantzig, G. B., Linear Programming and Extensions (Princeton, N. J.: Princeton University Press, 1963).

[7] Duffin, R. J., E. L. Peterson and C. Zener., Geometric Programming (New York: John Wiley & Sons, Inc., forthcoming).

[8] Graves, R. L. and P. Wolfe, eds., Recent Advances in Mathematical Programming (New York: McGraw-Hill Book Co., Inc., 1963).

[9] Muth, J. F., and G. L. Thompson, eds., Industrial Scheduling (Englewood Cliffs, N. J., 1963).

---

[10] Wilde, D. J. and M. Pasay, "Polynomial Inequalities: Geometric Programming," draft, Stanford University, Department of Chemical Engineering, 1966.

[11] Zener, C., "A Further Mathematical Aid in Optimizing Engineering Designs," Proc. Nat. Acad. Sci., 48, 1962, p. 518.

[12] _____ "A Mathematical Aid in Optimizing Engineering Designs," Proc. Nat. Acad. Sci. 47, 1961, pp. 537-539.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*
Graduate School of Industrial Administration
Carnegie Institute of Technology

2a. REPORT SECURITY CLASSIFICATION
Unclassified

2b. GROUP
Not applicable

3. REPORT TITLE
A CONVEX APPROXIMANT METHOD FOR NON-CONVEX EXTENSIONS OF GEOMETRIC PROGRAMMING

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
Technical Report, May 1966

5. AUTHOR(S) *(Last name, first name, initial)*
Charnes A., and Cooper, W. W.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May, 1966	8	12

8a. CONTRACT OR GRANT NO.
NONR 760(24)

b. PROJECT NO.
NR 047-048

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)
Management Sciences Research Report No. 76*

9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
Systems Research Memo. No. (see item 11)

10. AVAILABILITY/LIMITATION NOTICES
Releasable without limitations on dissemination

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
Also under Contract NONR 1228(10) Project NR 047-021 at Northwestern University	Logistics and Mathematical Statistics Branch Office of Naval Research Washington, D. C. 20360

13. ABSTRACT

Many important problems of engineering and management are of a form which could be represented as geometric programs except that the functional to be minimized as well as the constraints are not confined to posynomials in that some of the coefficients are negative. This paper supplies a way for dealing with such negative terms by a constraint adjunction procedure which yields an associated approximating problem involving only posynomials which can, in turn, be transformed into a convex programming problem that has only one local (= global) optimum. The latter, which is called a convex approximant, has an associated dual. Recourse to the related duality theory then supplies guidance for improving the approximation along lines that are indicated in the paper.

DD FORM 1 JAN 64 1473

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		INK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Geometric programming Convex programming Linear programming Duality theory Engineering designs Constrained inventory models Production function Approximation theory Capital budgets						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through

_____."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through

_____."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through

_____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.